

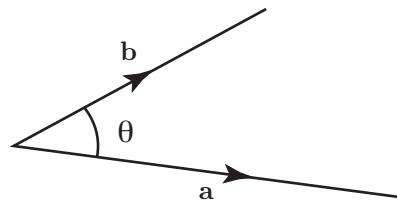
## The scalar product

### Introduction

On this leaflet we describe how to find the **scalar product** of two vectors.

### 1. Definition of the scalar product

Consider the two vectors **a** and **b** shown below. Note that the tails of the two vectors coincide and that the angle between the vectors has been labelled  $\theta$ .



Their scalar product, denoted  $\mathbf{a} \cdot \mathbf{b}$ , is defined as  $|\mathbf{a}| |\mathbf{b}| \cos \theta$ . It is very important to use the dot in the formula. The dot is the symbol for the scalar product, and is the reason why the scalar product is also known as the **dot product**. You should never use a  $\times$  sign in this context because this symbol is reserved for a quantity called the **vector product** which is quite different.

$$\text{scalar product : } \mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta$$

### Example

Vectors **a** and **b** are shown in the figure above. Suppose the vector **a** has modulus 8 and the vector **b** has modulus 7. Suppose also that the angle,  $\theta$ , between these vectors is  $30^\circ$ . Calculate  $\mathbf{a} \cdot \mathbf{b}$ .

### Solution

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= |\mathbf{a}| |\mathbf{b}| \cos \theta \\ &= (8)(7) \cos 30^\circ \\ &= 48.5 \end{aligned}$$

The scalar product of **a** and **b** is equal to 48.5. Note that when finding a scalar product the result is always a scalar, that is a number, and not a vector.

## 2. A formula for finding the scalar product

A simple formula exists for finding a scalar product when the vectors are given in cartesian form.

$$\text{if } \mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \quad \text{and} \quad \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k} \quad \text{then}$$
$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

### Example

If  $\mathbf{a} = 5\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$  and  $\mathbf{b} = 8\mathbf{i} - 9\mathbf{j} + 11\mathbf{k}$ , find  $\mathbf{a} \cdot \mathbf{b}$ .

### Solution

Respective components are multiplied together and the results are added.

$$\mathbf{a} \cdot \mathbf{b} = (5)(8) + (3)(-9) + (-2)(11) = 40 - 27 - 22 = -9$$

Note again that the result is a scalar not a vector. The answer cannot contain  $\mathbf{i}$ ,  $\mathbf{j}$ , or  $\mathbf{k}$ .

### Exercises

1. If  $\mathbf{a} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ ,  $\mathbf{b} = 7\mathbf{i} + \mathbf{j} + 2\mathbf{k}$  and  $\mathbf{c} = -\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$  show  $\mathbf{a} \cdot \mathbf{b} = 21$ ,  $\mathbf{b} \cdot \mathbf{c} = 1$  and  $\mathbf{a} \cdot \mathbf{c} = 8$ .

## 3. Using the scalar product to find the angle between two vectors

The scalar product is useful when you need to calculate the angle between two vectors.

### Example

Find the angle between the vectors  $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$  and  $\mathbf{b} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ .

### Solution

Their scalar product is easily shown to be 11. The modulus of  $\mathbf{a}$  is  $\sqrt{2^2 + 3^2 + 5^2} = \sqrt{38}$ . The modulus of  $\mathbf{b}$  is  $\sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{14}$ . Using the formula for the scalar product we find

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta$$
$$11 = \sqrt{38} \sqrt{14} \cos \theta$$

from which

$$\cos \theta = \frac{11}{\sqrt{38}\sqrt{14}} = 0.4769 \quad \text{so that} \quad \theta = \cos^{-1}(0.4769) = 61.5^\circ$$

In general, the angle between two vectors can be found from the following formula:

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|}$$

### Exercise

1. Show that the angle between the vectors  $\mathbf{a} = 5\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$  and  $\mathbf{b} = 8\mathbf{i} - 9\mathbf{j} + 11\mathbf{k}$  is  $95.14^\circ$ .